

# THE NIELSEN-THURSTON CLASSIFICATION OF MAPPING CLASSES IS DETERMINED BY TQFT

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ABSTRACT. For each fixed  $n \geq 2$  we show how the Nielsen-Thurston classification of mapping classes for a closed surface of genus  $g \geq 2$  is determined by the sequence of quantum  $SU(n)$ -representations  $(\rho_k)_{k \in \mathbb{N}}$ . That this is the case is a consequence of the asymptotic faithfulness property proved in [A3]. We here provide explicit conditions on  $(\rho_k(\phi))_{k \in \mathbb{N}}$ , which determines the Nielsen-Thurston type of any mapping class  $\phi$ .

## 1. INTRODUCTION

The Nielsen-Thurston classification of mapping classes of compact oriented surfaces splits mapping classes into three disjoint types [Th] (see also [FLP] and [BC]). We shall only be interested in the closed surface case here, so we state the Nielsen-Thurston theorem in this case.

Suppose  $\Sigma$  is a closed oriented surface of genus  $g \geq 2$  and let  $\Gamma$  be the mapping class group of  $\Sigma$ .

**Theorem 1** (Nielsen-Thurston). *A mapping class  $\phi \in \Gamma$  has exactly one of the following three properties.*

1. *The mapping class  $\phi$  is finite order, that is  $\phi$  is a finite order element in  $\Gamma$ . This is equivalent to  $\phi$  having an automorphism of a Riemann surface as representative.*
2. *The mapping class  $\phi$  is not finite order, but it is reducible, meaning there exists a simple closed curve on the surface, whose non-trivial homotopy class is preserved by some power of  $\phi$ .*
3. *The mapping class  $\phi$  is Pseudo-Anosov, meaning that there exists  $\lambda > 1$ , two transverse measured foliations  $F^s$  and  $F^u$  on  $\Sigma$  and a diffeomorphism  $f$  of  $\Sigma$ , which represents  $\phi$ , such that*

$$f_*(F^s) = \lambda^{-1}F^s \text{ and } f_*(F^u) = \lambda F^u.$$

In the Pseudo-Anosov case,  $\lambda$  is uniquely determined by  $\phi$  and it is called the stretching factor for  $\phi$ .

In the reducible case one continues the analysis of  $\phi$ , by cutting  $\Sigma$  along the preserved simple closed curve, to get a mapping class of a surface with boundary. This mapping class is then classified in terms of the Nielsen-Thurston classification of mapping classes of surfaces with boundary. The upshot of this is that there is a diffeomorphism  $f$  of  $\Sigma$ , which represents  $\phi$  and preserve a system of simple closed curves, and when one cuts the surface along these curves,  $f$  induces a diffeomorphism of the resulting components, which on each piece is either finite order or Pseudo-Anosov. See [FLP] for further details regarding this.

Fix an  $n \geq 2$  and  $d \in \mathbb{Z}/n\mathbb{Z}$ . Let  $\rho_k$  be the quantum  $SU(n)$  projective representations of the mapping class group  $\Gamma$  at level  $k$  (with twist  $d$  as described in [A3]), which arises from the Reshetikhin-Turaev  $SU(n)$  Topological Quantum Field Theory. These were first discussed by Witten in [W1]. Then they were rigorously constructed by Reshetikhin and Turaev in [RT1] and [RT2]. Subsequently they were constructed using skein theory in [BHMV1], [BHMV2] and [B1].

The objective of this paper is to provide explicit conditions on the endomorphisms  $\rho_k(\phi)$  which determines which Nielsen-Thurston type  $\phi \in \Gamma$  belongs to.

Recall that we proved the asymptotic faithfulness property of the sequence  $\rho_k$  in [A3]:

**Theorem 2.** *Assume that  $n$  and  $d$  are coprime or that  $(n, d) = (2, 0)$  when  $g = 2$ . Then we have that*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k) = \begin{cases} \{1, H\} & g = 2, n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise.} \end{cases}$$

where  $H$  is the hyperelliptic involution.

The non coprime cases is covered in [A4]<sup>1</sup>.

Logically it follows from this Theorem, that the Nielsen-Thurston classification of a mapping class  $\phi$  is determined by  $(\rho_k(\phi))_{k \in \mathbb{N}}$ , since this sequence determines  $\phi$  itself. - However, we would here like to provide explicit conditions on  $(\rho_k(\phi))_{k \in \mathbb{N}}$ , which separates the three Nielsen-Thurston types.

The asymptotic faithfulness property gives us immediately the following theorem.

**Theorem 3.** *For any mapping class  $\phi \in \Gamma$  we have that there exist an integer  $M$  such that*

$$(\rho_k(\phi))^M \in \mathbb{C} \text{Id}$$

for all  $k$  if and only if  $\phi^M = 1$  (or  $\phi^{2M} = 1$ , in case  $(n, d) = (0, 2)$ ).

This separates the finite order ones from the rest. In order to separate the reducibles from the Pseudo-Anosov, we recall the construction of the quantum representations and the Toeplitz operator construction from [A2] and [A3].

Let  $p$  a point on  $\Sigma$ . Let  $M$  be the moduli space of flat  $SU(n)$ -connections on  $\Sigma - p$  with holonomy  $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$  around  $p$ . Assume that  $n$  and  $d$  are coprime or that  $(n, d) = (2, 0)$  when  $g = 2$ . Then  $M$  is a smooth manifold.

By applying geometric quantization at level  $k$  to the moduli space  $M$  one gets a vector bundle  $\mathcal{H}^{(k)}$  over Teichmüller space  $\mathcal{T}$ . The fiber of this bundle over a point  $\sigma \in \mathcal{T}$  is  $\mathcal{H}_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}_\sigma^k)$ , where  $M_\sigma$  is  $M$  equipped with a complex structure induced from  $\sigma$  and  $\mathcal{L}_\sigma$  is an ample generator of the Picard group of  $M_\sigma$ .

The main result pertaining to this bundle  $\mathcal{H}^{(k)}$  is that its projectivization  $\mathbb{P}(\mathcal{H}^{(k)})$  supports a natural flat connection. This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. Now, since there is an action of the mapping class group  $\Gamma$  of  $\Sigma$  on  $\mathcal{H}^{(k)}$  covering its action on  $\mathcal{T}$ , which preserves the flat connection in  $\mathbb{P}(\mathcal{H}^{(k)})$ , we get for each  $k$  a finite dimensional projective representation, say  $\rho_k$ , of  $\Gamma$ , namely on the covariant constant sections of  $\mathbb{P}(\mathcal{H}^{(k)})$  over  $\mathcal{T}$ . This sequence of projective representations  $\rho_k$ ,  $k \in \mathbb{N}$  is the *quantum  $SU(n)$  representations* of the mapping class group  $\Gamma$ .

<sup>1</sup>In [FWW] our argument from [A3] was translated to the BHMV-skein model. See also [M2].

For each  $f \in C^\infty(M)$  and each point  $\sigma \in \mathcal{T}$ , we have the *Toeplitz operator*

$$T_{f,\sigma}^{(k)} : H^0(M_\sigma, \mathcal{L}_\sigma^k) \rightarrow H^0(M_\sigma, \mathcal{L}_\sigma^k)$$

which is given by

$$T_{f,\sigma}^{(k)} = \pi_\sigma^{(k)}(fs)$$

for all  $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$ , where  $\pi_\sigma^{(k)}$  is the orthogonal projection onto  $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ . We get smooth section of  $\text{End}(\mathcal{H}^{(k)})$  over  $\mathcal{T}$

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \text{End}(\mathcal{H}^{(k)}))$$

by letting  $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$  (see [A3]).

The  $L_2$ -inner product on  $C^\infty(M, \mathcal{L}^k)$  induces an inner product on  $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ , which in turn induces the operator norm  $\|\cdot\|$  on  $\text{End}(H^0(M_\sigma, \mathcal{L}_\sigma^k))$ . Hence for any  $A \in C^\infty(\mathcal{T}, \text{End}(\mathcal{H}^{(k)}))$  we get the smooth function  $\|A\|$  on  $\mathcal{T}$ .

Suppose now  $\gamma$  is a closed curve on the surface  $\Sigma$ . Then we have the holonomy function  $h_\gamma$  defined on  $M$  associated to  $\gamma$ , given by taking the trace of the holonomy around  $\gamma$ . Note that  $h_\gamma$  only depends on the free homotopy class of  $\gamma$ . Further  $h_\gamma$  is constant if  $\gamma$  is nul-homotopic.

**Theorem 4.** *For any mapping class  $\phi \in \Gamma$  and any homotopy class  $\gamma$  of a simple closed curve on  $\Sigma$  we have that  $\phi$  is reducible along  $\gamma$ , i.e.*

$$\phi(\gamma) = \gamma$$

*if and only if  $T_{h_\gamma}^{(k)}$  asymptotically commutes with  $\rho_k(\phi)$ :*

$$\lim_{k \rightarrow \infty} \|[\rho_k(\phi), T_{h_\gamma}^{(k)}]\| = 0.$$

Here the limit is pointwise convergence over  $\mathcal{T}$ . In fact, it follows from the results in [A3] that if the limit is zero at some point in  $\mathcal{T}$  then it holds for all points in  $\mathcal{T}$  and in fact the convergence is uniform on compact subsets of  $\mathcal{T}$ .

Using these two conditions, we see immediately how to determine the Nielsen-Thurston classification of a mapping class:

1. Given a mapping class  $\phi$ , we first determine if it is finite order or not by checking if  $\rho_k(\phi)$  is finite order bounded in  $k$ .
2. If not, we check through the non-trivial homotopy classes  $\gamma$  of simple closed curves to see if there exist an integer  $M$  with the property that  $T_{h_\gamma}^{(k)}$  asymptotically commutes with  $\rho_k(\phi^M)$ :

$$\lim_{k \rightarrow \infty} \|[\rho_k(\phi^M), T_{h_\gamma}^{(k)}]\| = 0.$$

If so,  $\phi$  is reducible.

3. If not, then  $\phi$  is Pseudo-Anosov.

We expect the following should be true.

**Conjecture 1** (AMU). *For a mapping class  $\phi \in \Gamma$  we have that  $\phi$  is Pseudo-Anosov or reducible with Pseudo-Anosov pieces if and only if  $\rho_k(\phi)$  is infinite order for large enough  $k$ .*

Evidence for this conjecture is provided by [M1] and [AMU].

This paper is organized as follows. In section 2 we recall the needed gauge theory setup. In the process we prove that the free homotopy class of a simple closed curve on a surface is determined by its holonomy function on the  $SU(n)$  moduli space (for each  $n \geq 2$ ). In section 3 we recall the main analytic estimate from [A3], which states that the Toeplitz operators are asymptotically flat with respect to Hitchin's connection. The proof of Theorem 4 is given in section 4. In the last section we translate the formulation of the results of this paper in the BHMV skein model for these TQFT's.

We would like to thank Gregor Masbaum, Bill Goldman and Peter Storm for discussions related to this work.

## 2. THE GAUGE THEORY CONSTRUCTION OF THE QUANTUM $SU(n)$ REPRESENTATIONS.

Let us now very briefly recall the construction of the quantum  $SU(n)$  representations. Only the details needed in this paper will be given. We refer e.g. to [H], [A3] and [A5] for further details. As in the introduction we let  $\Sigma$  be a closed oriented surface of genus  $g \geq 2$  and  $p \in \Sigma$ . Let  $P$  be a principal  $SU(n)$ -bundle over  $\Sigma$ . Clearly, all such  $P$  are trivializable. As above let  $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$ . Throughout the rest of this paper we will assume that  $n$  and  $d$  are coprime, although in the case  $g = 2$  we also allow  $(n, d) = (2, 0)$ . Let  $M$  be the moduli space of flat  $SU(n)$ -connections in  $P|_{\Sigma-p}$  with holonomy  $d$  around  $p$ . We can identify

$$M = \text{Hom}_d(\tilde{\pi}_1(\Sigma), SU(n))/SU(n).$$

Here  $\tilde{\pi}_1(\Sigma)$  is the universal central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{\pi}_1(\Sigma) \rightarrow \pi_1(\Sigma) \rightarrow 1$$

as discussed in [H] and in [AB] and  $\text{Hom}_d$  means the space of homomorphisms from  $\tilde{\pi}_1(\Sigma)$  to  $SU(n)$  which send the image of  $1 \in \mathbb{Z}$  in  $\tilde{\pi}_1(\Sigma)$  to  $d$  (see [H]). When  $n$  and  $d$  are coprime,  $M$  is a compact smooth manifold of dimension  $m = (n^2 - 1)(g - 1)$ . In general, when  $n$  and  $d$  are not coprime  $M$  is not smooth, except in the case where  $g = 2$ ,  $n = 2$  and  $d = 0$ . In this case  $M$  is in fact diffeomorphic to  $\mathbb{CP}^3$ . There is a natural homomorphism from the mapping class group to the outer automorphisms of  $\tilde{\pi}_1(\Sigma)$ , hence  $\Gamma$  acts on  $M$ .

We have that  $M$  is a component of the real slice in  $\mathcal{M}$ , the moduli space of flat  $SL(n, \mathbb{C})$ -connections on  $\Sigma - p$ , whose holonomy around  $p$  has trace  $n \exp(2\pi\sqrt{-1}\frac{d}{n})$ . This space is connected and it contains the generalized Teichmüller space  $\tilde{\mathcal{T}}_p$  of  $\Sigma - p$ .

Suppose  $\gamma$  is a closed curve on  $\Sigma - p$ . Then we have the algebraic holonomy function  $h_\gamma$  defined on the variety  $\mathcal{M}$ . We are in particular interested in the restriction of  $h_\gamma$  to  $M$ , where it is a smooth function. We observe that this restriction map is injective, since  $M$  is a real slice in  $\mathcal{M}$ , which is connected.

**Proposition 1.** *If  $\gamma$  is a simple closed curve on  $\Sigma - p$ , then the holonomy function  $h_\gamma$  restricted to  $M$  determines  $\gamma$  up to free homotopy on  $\Sigma$ .*

*Proof.* Suppose we have two simple closed curves  $\gamma_1$  and  $\gamma_2$ , such that  $h_{\gamma_1} = h_{\gamma_2}$ . Then if the geometric intersection number of the homotopy classes of the two curves is positive, we see that the two functions cannot possibly be identical. This follows from considering the functions restricted to the classical Teichmüller space for the punctured surface  $\Sigma - p$ . Consider a sequence of metrics where the length of say

$\gamma_1$  goes to zero. Then the length of  $\gamma_2$  must go to infinity. But the length of  $\gamma_i$  is determined by  $h_{\gamma_i}$ , so we get a contradiction.

Now suppose the geometric intersection number of the homotopy classes of the two curves is zero. Then we can vary the holonomy along the two curves completely independently, if they are not freely homotopic. Hence they must be freely homotopic on  $\Sigma$ .

□

Continuing towards the gauge theory definition of the quantum  $SU(n)$  representations, we now choose an invariant bilinear form  $\{\cdot, \cdot\}$  on  $\mathfrak{g} = \text{Lie}(SU(n))$ , normalized such that  $-\frac{1}{6}\{\vartheta \wedge [\vartheta \wedge \vartheta]\}$  is a generator of the image of the integer cohomology in the real cohomology in degree 3 of  $SU(n)$ , where  $\vartheta$  is the  $\mathfrak{g}$ -valued Maurer-Cartan 1-form on  $SU(n)$ .

This bilinear form induces a symplectic form on  $M$ . In fact

$$T_{[A]}M \cong H^1(\Sigma, d_A),$$

where  $A$  is any flat connection in  $P$  representing a point in  $M$  and  $d_A$  is the induced covariant derivative in the associated adjoint bundle. Using this identification, the symplectic form on  $M$  is:

$$\omega(\varphi_1, \varphi_2) = \int_{\Sigma} \{\varphi_1 \wedge \varphi_2\},$$

where  $\varphi_i$  are  $d_A$ -closed 1-forms on  $\Sigma$  with values in  $\text{ad } P$ . See e.g. [H] for further details on this. The natural action of  $\Gamma$  on  $M$  is symplectic.

Let  $\mathcal{L}$  be the Hermitian line bundle over  $M$  and  $\nabla$  the compatible connection in  $\mathcal{L}$  constructed by Freed in [Fr]. This is the content of Corollary 5.22, Proposition 5.24 and equation (5.26) in [Fr] (see also the work of Ramadas, Singer and Weitsman [RSW]). By Proposition 5.27 in [Fr] we have that the curvature of  $\nabla$  is  $\frac{\sqrt{-1}}{2\pi}\omega$ . We will also use the notation  $\nabla$  for the induced connection in  $\mathcal{L}^k$ , where  $k$  is any integer.

By an almost identical construction, we can lift the action of  $\Gamma$  on  $M$  to act on  $\mathcal{L}$  such that the Hermitian connection is preserved (See e.g. [A1]). In fact, since  $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$  and  $H^1(M, \mathbb{Z}) = 0$ , it is clear that the action of  $\Gamma$  leaves the isomorphism class of  $(\mathcal{L}, \nabla)$  invariant, thus alone from this one can conclude that a central extension of  $\Gamma$  acts on  $(\mathcal{L}, \nabla)$  covering the  $\Gamma$  action on  $M$ . This is actually all we need in this paper, since we are only interested in the projectivized action.

Let now  $\sigma \in \mathcal{T}$  be a complex structure on  $\Sigma$ . Let us review how  $\sigma$  induces a complex structure on  $M$  which is compatible with the symplectic structure on this moduli space. The complex structure  $\sigma$  induces a  $*$ -operator on 1-forms on  $\Sigma$  and via Hodge theory we get that

$$H^1(\Sigma, d_A) \cong \ker(d_A + *d_A*).$$

On this kernel, consisting of the harmonic 1-forms with values in  $\text{ad } P$ , the  $*$ -operator acts and its square is  $-1$ , hence we get an almost complex structure on  $M$  by letting  $I = I_{\sigma} = -*$ . It is a classical result by Narasimhan and Seshadri (see [NS1]), that this actually makes  $M$  a smooth Kähler manifold, which as such, we denote  $M_{\sigma}$ . By using the  $(0, 1)$  part of  $\nabla$  in  $\mathcal{L}$ , we get an induced holomorphic structure in the bundle  $\mathcal{L}$ . The resulting holomorphic line bundle will be denoted  $\mathcal{L}_{\sigma}$ . See also [H] for further details on this.

From a more algebraic geometric point of view, we consider the moduli space of  $S$ -equivalence classes of semi-stable bundles of rank  $n$  and determinant isomorphic to the line bundle  $\mathcal{O}(d[p])$ . By using Mumford's Geometric Invariant Theory, Narasimhan and Seshadri (see [NS2]) showed that this moduli space is a smooth complex algebraic projective variety which is isomorphic as a Kähler manifold to  $M_\sigma$ . Referring to [DN] we recall that

**Theorem 5** (Drezet & Narasimhan). *The Picard group of  $M_\sigma$  is generated by the holomorphic line bundle  $\mathcal{L}_\sigma$  over  $M_\sigma$  constructed above:*

$$\text{Pic}(M_\sigma) = \langle \mathcal{L}_\sigma \rangle.$$

**Definition 1.** *The bundle  $\mathcal{H}^{(k)}$  over Teichmüller space is by definition the bundle whose fiber over  $\sigma \in \mathcal{T}$  is  $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ , where  $k$  is a positive integer.*

There is a projective flat connection  $\hat{\nabla}$  in the bundle  $\mathcal{H}^{(k)}$  over Teichmüller space, which is  $\Gamma$ -invariant due to Axelrod, Della Pietra and Witten's and Hitchin's. We refer to [H] and [ADW] for further details. See also [A3] and [A5]. Faltings has established this theorem in the case where one replaces  $SU(n)$  with a general semi-simple Lie group (see [Fal]). We further remark about genus 2, that [ADW] covers this case, but [H] excludes this case, however, the work of Van Geemen and De Jong [vGdJ] extends Hitchin's approach to the genus 2 case.

**Definition 2.** *The quantum  $SU(n)$  projective representation  $\rho_k$  at level  $k$  of the mapping class group is given by  $\Gamma$ 's action on the covariant constant sections of  $(\mathbb{P}(\mathcal{H}^{(k)}), \hat{\nabla})$ . We use the notation  $\mathbb{P}(Z^{(k)}(\Sigma))$  for this projective space of covariant constant sections.*

We denote the induced flat connection in the endomorphism bundle  $\text{End}(\mathcal{H}^{(k)})$  by  $\hat{\nabla}^e$ .

**Definition 3.** *Let  $\mathcal{E}^{(k)}(\Sigma)$  be the finite dimensional algebra of covariant constant sections of  $(\text{End}(\mathcal{H}^{(k)}), \hat{\nabla}^e)$  over  $\mathcal{T}$  and let*

$$\rho_k^e : \Gamma \rightarrow \text{Aut}(\mathcal{E}^{(k)})$$

*be the corresponding representation.*

### 3. TOEPLITZ OPERATORS AND THEIR ASYMPTOTIC FLATNESS

On  $C^\infty(M, \mathcal{L}^k)$  we have the  $L_2$ -inner product:

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^m$$

where  $s_1, s_2 \in C^\infty(M, \mathcal{L}^k)$  and  $(\cdot, \cdot)$  is the fiberwise Hermitian structure in  $\mathcal{L}^k$ .

Now let  $\sigma \in \mathcal{T}$ . Then this  $L_2$ -inner product gives the orthogonal projection

$$\pi_\sigma^{(k)} : C^\infty(M, \mathcal{L}^k) \rightarrow H^0(M_\sigma, \mathcal{L}_\sigma^k).$$

For each  $f \in C^\infty(M)$  consider the associated *Toeplitz operator*  $T_{f,\sigma}^{(k)}$  given as the composition of the multiplication operator

$$M_f : H^0(M_\sigma, \mathcal{L}_\sigma^k) \rightarrow C^\infty(M, \mathcal{L}^k)$$

with the orthogonal projection:

$$T_{f,\sigma}^{(k)}(s) = \pi_\sigma^{(k)}(fs).$$

Then  $T_{f,\sigma}^{(k)} \in \text{End}(H^0(M_\sigma, \mathcal{L}_\sigma^k))$ , and we get a smooth section

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \text{End}(\mathcal{H}^{(k)}))$$

by letting  $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$  (see [A3]).

The  $L_2$ -inner product on  $C^\infty(M, \mathcal{L}^k)$  induces an inner product on  $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ , which in turn induces the operator norm  $\|\cdot\|$  on  $\text{End}(H^0(M_\sigma, \mathcal{L}_\sigma^k))$ .

We need the following Theorem on Toeplitz operators due to Bordemann, Meinrenken and Schlichenmaier (see [BMS], [Sch], [Sch1] and [Sch2]).

**Theorem 6** (Bordemann, Meinrenken and Schlichenmaier). *For any  $f \in C^\infty(M)$  we have that*

$$\lim_{k \rightarrow \infty} \|T_{f,\sigma}^{(k)}\| = \sup_{x \in M} |f(x)|.$$

Since the association of the sequence of Toeplitz operators  $T_{f,\sigma}^k$ ,  $k \in \mathbb{Z}_+$  is linear in  $f$ , we see from this Theorem, that this association is faithful.

Theorem 10 in [A3] tells us that the Toeplitz operators are asymptotically flat with respect to  $\hat{\nabla}^e$ :

**Theorem 7.** *Let  $\sigma_0$  and  $\sigma_1$  be two points in Teichmüller space and  $P_{\sigma_0, \sigma_1}$  be the parallel transport in the flat bundle  $(\text{End}(\mathcal{H}^{(k)}), \hat{\nabla}^e)$  from  $\sigma_0$  to  $\sigma_1$ . Then*

$$\|P_{\sigma_0, \sigma_1} T_{f, \sigma_0}^{(k)} - T_{f, \sigma_1}^{(k)}\| = O(k^{-1}),$$

where  $\|\cdot\|$  is the operator norm on  $H^0(M_{\sigma_1}, \mathcal{L}_{\sigma_1}^k)$ .

#### 4. PROOF OF THEOREM 4

First we establish the following Proposition.

**Proposition 2.** *For any  $\phi \in \Gamma$ ,  $f \in C^\infty(M)$  and  $\sigma \in \mathcal{T}$ , we have that*

$$\lim_{k \rightarrow \infty} \|T_{f,\sigma}^{(k)} - \rho_k(\phi) T_{f,\sigma}^{(k)} \rho_k(\phi^{-1})\| = \lim_{k \rightarrow \infty} \|T_{(f-f \circ \phi), \sigma}^{(k)}\|.$$

*Proof.* Suppose we have a  $\phi \in \Gamma$ . Then  $\phi$  induces a symplectomorphism of  $M$  which we also just denote  $\phi$  and we get the following commutative diagram for any  $f \in C^\infty(M)$

$$\begin{array}{ccccc} H^0(M_\sigma, \mathcal{L}_\sigma^k) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}_{\phi(\sigma)}^k) & \xrightarrow{P_{\phi(\sigma), \sigma}} & H^0(M_\sigma, \mathcal{L}_\sigma^k) \\ T_{f,\sigma}^{(k)} \downarrow & & T_{f \circ \phi, \phi(\sigma)}^{(k)} \downarrow & & \downarrow P_{\phi(\sigma), \sigma} T_{f \circ \phi, \phi(\sigma)}^{(k)} \\ H^0(M_\sigma, \mathcal{L}_\sigma^k) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}_{\phi(\sigma)}^k) & \xrightarrow{P_{\phi(\sigma), \sigma}} & H^0(M_\sigma, \mathcal{L}_\sigma^k), \end{array}$$

where  $P_{\phi(\sigma), \sigma} : H^0(M_{\phi(\sigma)}, \mathcal{L}_{\phi(\sigma)}^k) \rightarrow H^0(M_\sigma, \mathcal{L}_\sigma^k)$  on the horizontal arrows refer to parallel transport in the bundle  $\mathcal{H}^{(k)}$ , whereas  $P_{\phi(\sigma), \sigma}$  refers to the parallel transport in the endomorphism bundle  $\text{End}(\mathcal{H}^{(k)})$  in the last vertical arrow. By the definition of  $\rho_k$ , we see that

$$\rho_k(\phi) T_{f,\sigma}^{(k)} \rho_k(\phi^{-1}) = P_{\phi(\sigma), \sigma} T_{f \circ \phi, \phi(\sigma)}^{(k)}.$$

By Theorem 7 we get that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|T_{(f-f \circ \phi), \sigma}^{(k)}\| &= \lim_{k \rightarrow \infty} \|T_{f, \sigma}^{(k)} - T_{f \circ \phi, \sigma}^{(k)}\| \\ &= \lim_{k \rightarrow \infty} \|T_{f, \sigma}^{(k)} - P_{\phi(\sigma), \sigma} T_{f \circ \phi, \phi(\sigma)}^{(k)}\| \\ &= \lim_{k \rightarrow \infty} \|T_{f, \sigma}^{(k)} - \rho_k(\phi) T_{f, \sigma}^{(k)} \rho_k(\phi^{-1})\|. \end{aligned}$$

□

*Proof of Theorem 4.* Let  $\phi \in \Gamma$  and assume that there exists  $\gamma$  non nul-homotopic simple closed curve on  $\Sigma$  and  $\sigma \in \mathcal{T}$ , such that

$$\lim_{k \rightarrow \infty} \|[\rho_k(\phi), T_{h_\gamma, \sigma}^{(k)}]\| = 0.$$

But

$$\begin{aligned} \|T_{h_\gamma, \sigma}^{(k)} - \rho_k(\phi^{-1}) T_{h_\gamma, \sigma}^{(k)} \rho_k(\phi)\| &= \|\rho_k(\phi^{-1}) \rho_k(\phi) (T_{h_\gamma, \sigma}^{(k)} - \rho_k(\phi^{-1}) T_{h_\gamma, \sigma}^{(k)} \rho_k(\phi))\| \\ &\leq \|\rho_k(\phi^{-1})\| \|[\rho_k(\phi), T_{h_\gamma, \sigma}^{(k)}]\|. \end{aligned}$$

Lemma 1 and Proposition 2 in [A3] gives a uniform bound on  $\|\rho_k(\phi^{-1})\|$ . Hence Proposition 2 implies that

$$\lim_{k \rightarrow \infty} \|T_{(h_\gamma - h_\gamma \circ \phi^{-1}), \sigma}^{(k)}\| = 0.$$

Then by Bordemann, Meinrenken and Schlichenmaier's Theorem 6, we must have that  $h_\gamma = h_\gamma \circ \phi^{-1}$ . But so by Proposition 1, we must have that  $\phi$  preserved the homotopy class of  $\gamma$ .

The converse follows directly from Proposition 2, since  $h_\gamma = h_\gamma \circ \phi$ , if  $\phi(\gamma) = \gamma$ .

□

## 5. THE FORMULATION IN THE BHMV-SKEIN MODEL

In this section we provide a translation of our conditions for separating the Nielsen-Thurston types of mapping classes to the BHMV-skein model. The determination of the finite order elements, Theorem 3 needs no translation, since it follows directly from the asymptotic faithfulness property. We just need to translate the condition in Theorem 4 which determines when a mapping class is reducible.

Let  $V_p$  be the TQFT defined by Blanchet, Habegger, Masbaum and Vogel in [BHMV1] and [BHMV2], where  $p = 2r$ ,  $r$  being an integer and we choose the  $2p$  root of 1 to be  $A = e^{i\pi/2r}$ . In particular there is also a projective representation

$$V_p : \Gamma \rightarrow \text{Aut}(\mathbb{P}(V_p(\Sigma)))$$

for each  $p$ .

Let  $\mathcal{S}(\Sigma)$  be the free  $\mathbb{C}$ -vector space generated by isotopy classes of one-dimensional sub-manifolds of  $\Sigma$ . We note that  $\mathcal{S}(\Sigma)$  is a subspace of the free  $\mathbb{C}$ -vector space generated by links in  $\Sigma \times [0, 1]$ .

Since  $V_p(\partial(\Sigma \times [0, 1])) = \text{End}(V_p(\Sigma))$ , we get by the very construction of the BHMV-skein model of this TQFT a sequence of linear maps

$$V_p : \mathcal{S}(\Sigma) \rightarrow \text{End}(V_p(\Sigma)).$$



**Theorem 8.** *For any mapping class  $\phi \in \Gamma$  and any non-trivial homotopy class  $\gamma$  of a simple closed curve on  $\Sigma$  we have that  $\phi$  is reducible along  $\gamma$ , i.e.*

$$\phi(\gamma) = \gamma$$

*if and only if*

$$[V_p(\phi), V_p(\gamma)] = 0$$

*for all  $p = 2r$ .*

Our joint work with K. Ueno [AU1], [AU2] and [AU3], combined with the work of Laszlo [La1] gives the following result for the  $(n, d) = (2, 0)$  theory:

**Theorem 9.** *There is a projective linear isomorphism of representations of  $\Gamma$*

$$I_k : \mathbb{P}(V_{2k+4}(\Sigma)) \rightarrow \mathbb{P}(Z^{(k)}(\Sigma)).$$

The projective linear isomorphism  $I_k$  induces a linear isomorphism of representations of  $\Gamma$

$$I_k^e : \text{End}(V_{2k+4}(\Sigma)) \rightarrow \mathcal{E}^{(k)}(\Sigma).$$

In [A6] we prove that

**Theorem 10.** *For any non-trivial homotopy class  $\gamma$  of a simple closed curve on  $\Sigma$  we have that*

$$\lim_{k \rightarrow \infty} \|I_k^e(V_{2k+4}(\gamma)) - T_{h_\gamma}^{(k)}\| = 0.$$

*Proof of Theorem 8.* Let  $\phi \in \Gamma$  be a mapping class. Assume there exists  $\gamma$  non nul-homotopic simple closed curve on  $\Sigma$  such that

$$[V_{2k+4}(\phi), V_{2k+4}(\gamma)] = 0$$

for all  $k \in \mathbb{N}$ . Then it follows immediately that

$$\lim_{k \rightarrow \infty} \|[I_k^e(V_{2k+4}(\phi)), T_{h_\gamma}^{(k)}]\| = 0$$

by Theorem 10. But since  $I_k^e(V_{2k+4}(\phi)) = \rho_k(\phi)$  by Theorem 9, we get by Theorem 4 that  $\phi$  is reducible. The converse statement is trivial.

□

After the completion of this work we where made aware of [MN]. One can translate our proof of Theorem 4 into a BHMV-skein model proof of Theorem 8 using the result of [MN].

## REFERENCES

- [A1] J.E. Andersen, "Jones-Witten theory and the Thurston boundary of Teichmüller space.", University of Oxford D. Phil thesis (1992), 129pp.
- [A2] J.E. Andersen, "Geometric quantization and deformation quantization of abelian moduli spaces.", Commun. Math. Phys. 255 (2005), 27 – 745.
- [A3] J.E. Andersen, "Asymptotic faithfulness of the quantum  $SU(n)$  representations of the mapping class groups", Annals of Mathematics **163** (2006), 347 – 368.
- [A4] J.E. Andersen, "Asymptotic faithfulness of the quantum  $SU(n)$  representations of the mapping class groups in the singular case", In preparation.
- [A5] J.E. Andersen, "Hitchin's connection, Toeplitz operators and symmetry invariant deformation quantization", Preprint.
- [A6] J.E. Andersen, "Asymptotic in Teichmüller space of the Hitchin connection", In preparation.

- [AMU] J.E. Andersen, G. Masbaum & K. Ueno, "Topological quantum field theory and the Nielsen-Thurston classification of  $M(0,4)$ ", math.GT/0503414. To appear in Math. Proc. Camb. Phil. Soc.
- [AU1] J.E. Andersen & K. Ueno, "Abelian Conformal Field theories and Determinant Bundles.", Preprint. math.QA/0304135.
- [AU2] J.E. Andersen & K. Ueno, "Constructing modular functors from conformal field theories.", Preprint. math.DG/0306235.
- [AU3] J.E. Andersen & K. Ueno, "Construction of the Reshetikhin-Turaev TQFT from conformal field theory.", In preparation.
- [At2] M. Atiyah, "The Jones-Witten invariants of knots." Séminaire Bourbaki, Vol. 1989/90. Astérisque No. **189-190** (1990), Exp. No. 715, 7–16.
- [AB] M. Atiyah & R. Bott, "The Yang-Mills equations over Riemann surfaces" Phil. Trans. R. Soc. Lond., Vol. **A308** (1982) 523 – 615.
- [ADW] S. Axelrod, S. Della Pietra, E. Witten, "Geometric quantization of Chern Simons gauge theory.", J.Diff.Geom. **33** (1991) 787–902.
- [B1] C. Blanchet, "Hecke algebras, modular categories and 3-manifolds quantum invariants.", Topology **39** (2000), no. 1, 193–223.
- [BHMV1] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, "Three-manifold invariants derived from the Kauffman Bracket." Topology **31** (1992), 685–699.
- [BHMV2] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, "Topological Quantum Field Theories derived from the Kauffman bracket." Topology **34** (1995), 883–927.
- [BC] S. Bleiler & A. Casson, "Automorphisms of surfaces after Nielsen and Thurston", Cambridge University Press, 1988.
- [BMS] M. Bordeman, E. Meinrenken & M. Schlichenmaier, "Toeplitz quantization of Kähler manifolds and  $gl(N)$ ,  $N \rightarrow \infty$  limit", Comm. Math. Phys. **165** (1994), 281–296.
- [DN] J.-M. Drezet & M.S. Narasimhan, "Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques", Invent. math. **97** (1989) 53–94.
- [Fal] G. Faltings, "Stable G-bundles and projective connections.", J.Alg.Geom. **2** (1993) 507–568.
- [FLP] A. Fathi, F. Laudenbach & V. Poénaru, "Travaux de thurston sur les surfaces", Astérisque **66-67** (1991/1979).
- [Fr] D.S. Freed, "Classical Chern-Simons Theory, Part 1", Adv. Math. **113** (1995), 237–303.
- [FWW] M. H. Freedman, K. Walker & Z. Wang, "Quantum SU(2) faithfully detects mapping class groups modulo center." Geom. Topol. **6** (2002), 523–539
- [vGdJ] B. Van Geemen & A. J. De Jong, "On Hitchin's connection", J. of Amer. Math. Soc., **11** (1998), 189–228.
- [H] N. Hitchin, "Flat connections and geometric quantization.", Comm.Math.Phys., **131** (1990) 347–380.
- [KS] A. V. Karabegov & M. Schlichenmaier, "Identification of Berezin-Toeplitz deformation quantization", J. Reine Angew. Math. **540** (2001), 49–76.
- [La1] Y. Laszlo, "Hitchin's and WZW connections are the same.", J. Diff. Geom. **49** (1998), no. 3, 547–576.
- [MN] J. Marché & M. Narimannejad, "Some asymptotics of TQFT via skein theory", math.GT/0604533 v1, 25 April 2006.
- [M1] G. Masbaum, "An element of infinite order in TQFT-representations of mapping class groups." Low-dimensional topology (Funchal, 1998), 137–139, Contemp. Math., **233**, Amer. Math. Soc., Providence, RI, 1999.
- [M2] G. Masbaum. Quantum representations of mapping class groups. In: Groupes et Géométrie (Journée annuelle 2003 de la SMF). pages 19–36
- [NS1] M.S. Narasimhan and C.S. Seshadri, "Holomorphic vector bundles on a compact Riemann surface", Math. Ann. **155** (1964) 69 – 80.
- [NS2] M.S. Narasimhan and C.S. Seshadri, "Stable and unitary vector bundles on a compact Riemann surface", Ann. Math. **82** (1965) 540 – 67.
- [R1] T.R. Ramadas, "Chern-Simons gauge theory and projectively flat vector bundles on  $M_g$ ", Comm. Math. Phys. **128** (1990), no. 2, 421 – 426.
- [RSW] T.R. Ramadas, I.M. Singer and J. Weitsman, "Some Comments on Chern - Simons Gauge Theory", Comm. Math. Phys. **126** (1989) 409–420.

- [RT1] N. Reshetikhin & V. Turaev, "Ribbon graphs and their invariants derived from quantum groups", *Comm. Math. Phys.* **127** (1990), 1 – 26.
- [RT2] N. Reshetikhin & V. Turaev, "Invariants of 3-manifolds via link polynomials and quantum groups", *Invent. Math.* **103** (1991), 547-597.
- [Ro] J. Roberts, "Irreducibility of some quantum representations of mapping class groups". *J. Knot Theory and its Ramifications* 10 (2001) 763 – 767.
- [Sch] M. Schlichenmaier, "Berezin-Toeplitz quantization and conformal field theory" Thesis.
- [Sch1] M. Schlichenmaier, "Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization." In *Conférence Moshé Flato 1999, Vol. II (Dijon)*, 289–306, *Math. Phys. Stud.*, **22**, Kluwer Acad. Publ., Dordrecht, (2000), 289–306.
- [Sch2] M. Schlichenmaier, "Berezin-Toeplitz quantization and Berezin transform." In *Long time behaviour of classical and quantum systems (Bologna, 1999)*, Ser. Concr. Appl. Math., **1**, World Sci. Publishing, River Edge, NJ, (2001), 271–287.
- [Th] W. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces", *Bull. of Amer. Math. Soc.* 19 (1988), 417 – 431.
- [TUY] A. Tsuchiya, K. Ueno & Y. Yamada, "Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries", *Advanced Studies in Pure Mathematics*, **19** (1989), 459–566.
- [T] V. G. Turaev, "Quantum invariants of knots and 3-manifolds." *de Gruyter Studies in Mathematics*, 18. Walter de Gruyter & Co., Berlin, 1994. x+588 pp. ISBN: 3-11-013704-6
- [W1] E. Witten, "Quantum field theory and the Jones polynomial.", *Commun. Math. Phys.* **121** (1989) 351–98.

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